§3.4 二维 r.v. 函数的分布

问题

$r.v.(X,Y)$ ，

$g(x, y)$ ，

$$Z = g(X,Y)$$

方法

(X,Y)

(X, Y) $r.v.$, Z

$$Z = z_k = g(x_{i_k}, y_{j_k})$$

$$P(Z = z_k) = \sum_{g(x_{i_k}, y_{j_k}) = z_k} P(X = x_{i_k}, Y = y_{j_k}) \quad k = 1, 2, \cdots$$

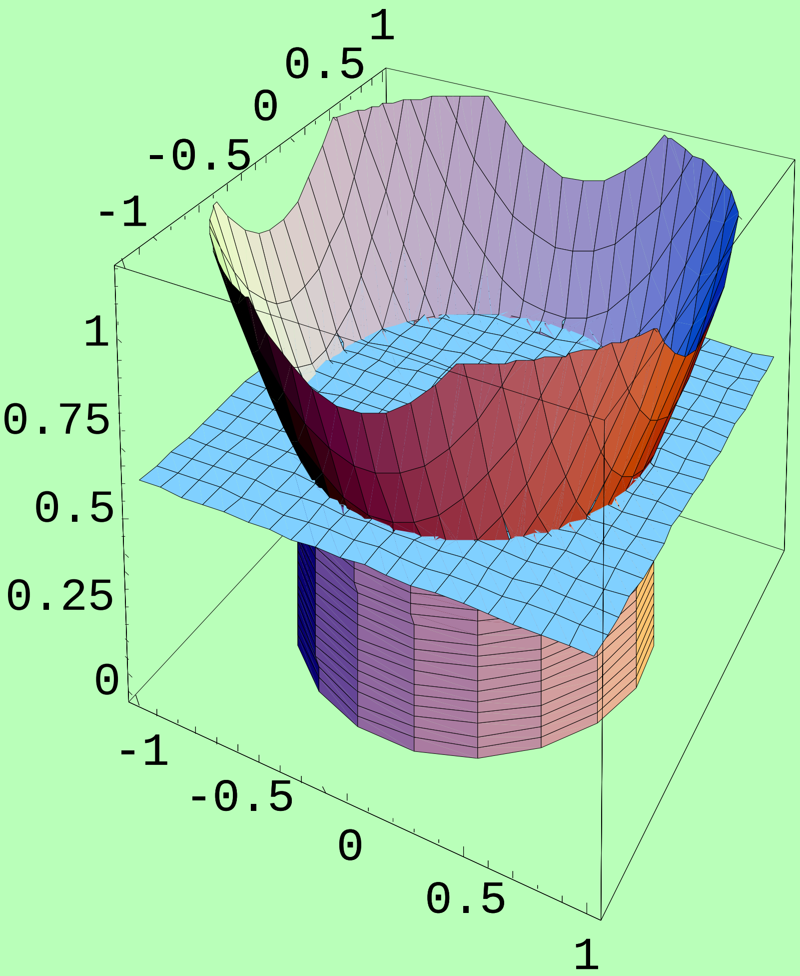
(X, Y) $r.v.$

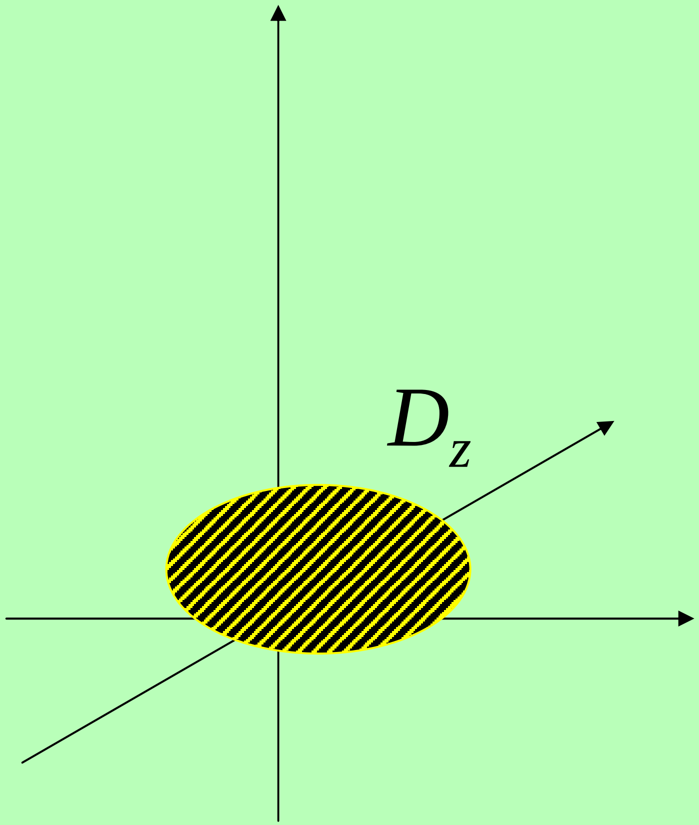
$$F_Z(z) = P(Z \le z) = P(g(X, Y) \le z)$$

$$= \iint_{D_z} f(x, y)\,dxdy$$

$$D_z : \{(x, y) \mid g(x, y) \le z\}$$

$$D_z : \{(x, y) \mid g(x, y) \leq z\}$$





$r.v.$

例1 $r.v.(X,Y)$

Y \ X (p_{ij})	-1	1	2
-1	$1/4$	$1/6$	$1/8$
0	$1/4$	$1/8$	$1/12$

$$X+Y, X-Y, XY, Y/X$$

解 (X,Y)

P	1/4	1/4	1/6	1/8	1/8	1/12
(X,Y)	(-1,-1)	(-1,0)	(1,-1)	(1,0)	(2,-1)	(2,0)
$X+Y$	-2	-1	0	1	1	2
$X-Y$	0	-1	2	1	3	2
XY	1	0	-1	0	-2	0
Y/X	1	0	-1	0	-1/2	0

$X+Y$	-2	-1	0	1	2
P	$1/4$	$1/4$	$1/6$	$1/4$	$1/12$

$X-Y$	-1	0	1	2	3
P	$1/4$	$1/4$	$1/8$	$1/4$	$1/8$

$X\ Y$	-2	-1	0	1
P	1/8	1/6	11/24	1/4

Y/X	-1	-1/2	0	1
P	1/6	1/8	11/24	1/4

具有可加性的两个离散分布

- $X \sim B(n_1, p),\ Y \sim B(n_2, p),$

  $X + Y \sim B(n_1 + n_2, p)$

- $X \sim P(\lambda_1),\ Y \sim P(\lambda_2),$

  $X + Y \sim P(\lambda_1 + \lambda_2)$

Poisson分布可加性的证明

$X \sim P(\lambda_1),\ Y \sim P(\lambda_2),$

$Z = X + Y$ 0,1,2, …,

$$P(Z=k)=\sum_{i=0}^{k}P(X=i,Y=k-i),$$

$$=\sum_{i=0}^{k}\frac{\lambda_1^i e^{-\lambda_1}}{i!}\cdot\frac{\lambda_2^{k-i}e^{-\lambda_2}}{(k-i)!}$$

$$=\frac{e^{-\lambda_1-\lambda_2}}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\lambda_1^i\lambda_2^{k-i}$$

$$=\frac{(\lambda_1+\lambda_2)^k e^{-\lambda_1-\lambda_2}}{k!} \qquad k=0,1,2,\cdots$$

r.v.

问题 $r.v.(X,Y)$ $d.f.$

$g(x,y)$

$Z = g(X,Y)$ $d.f.$

方法

- Z , Z
  (X,Y)
- $r.v.(Z,X)$ (Z,Y),
  Z $d.f.$

(1) $Z = X + Y$

(X, Y) d.f. $f(x,y)$,

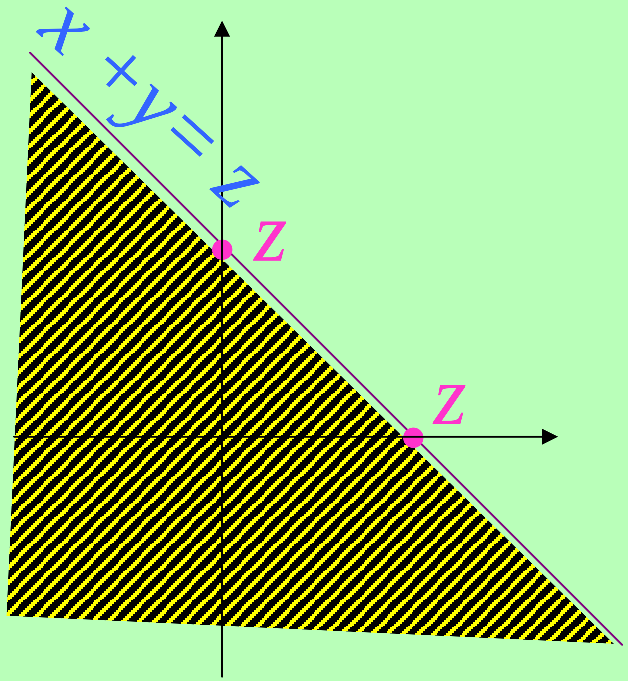
$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \iint_{x+y\le z} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z-x} f(x,y)\,dy$$

$$= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{z-y} f(x,y)\,dx \qquad -\infty < z < +\infty$$



$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx \quad -\infty < z < +\infty \quad (1)$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f(z-y, y)dy \quad -\infty < z < +\infty \quad (2)$$

X, Y

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx = f_X(z) * f_Y(z)$$
$$-\infty < z < +\infty \quad (3)$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y)dy = f_X(z) * f_Y(z)$$
$$-\infty < z < +\infty \quad (4)$$

$f_X(z)$ $f_Y(z)$ 卷积

例2 (X, Y) $d.f.$

$$f(x,y)=\begin{cases}1, & 0<x<1, 0<y<1\\ 0, & \end{cases}$$

$Z = X + Y$, $f_Z(z)$

解法一

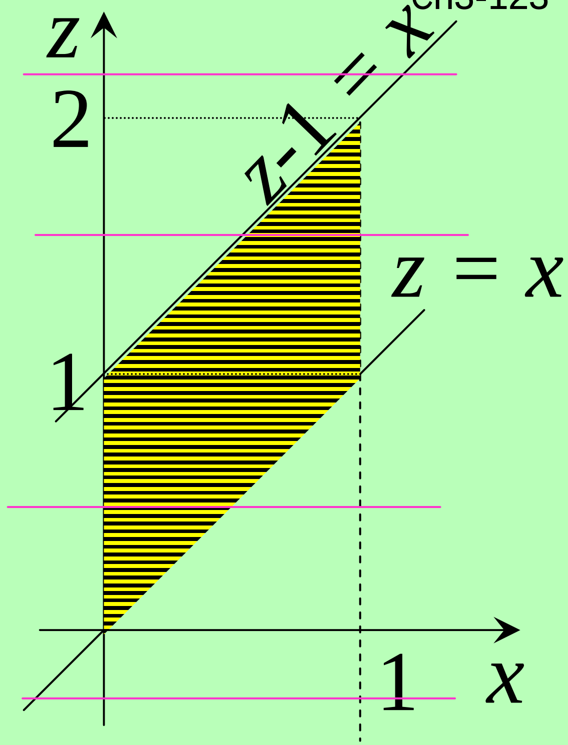
X, Y ,

$$f_X(x)=\begin{cases}1, & 0<x<1\\ 0, & \end{cases} \qquad f_Y(y)=\begin{cases}1, & 0<y<1\\ 0, & \end{cases}$$

$$f_Z(z)=\int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx$$

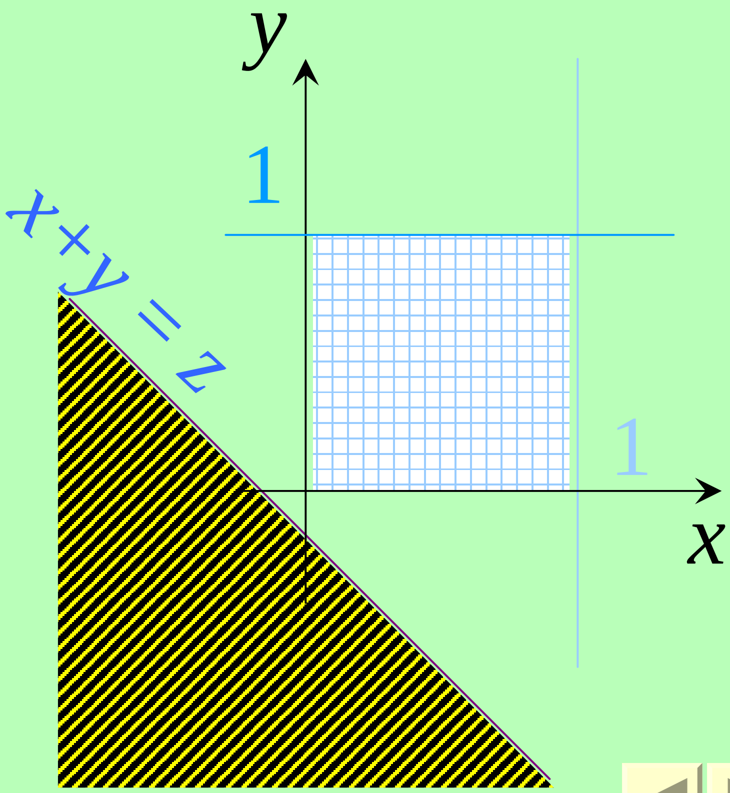
$$=\int_0^1 f_Y(z-x)dx$$

$$f_Y(z-x)=\begin{cases}1, & z-1<x<z\\ 0, & \end{cases}$$



$$\int_0^1 f_Y(z-x)dx=\begin{cases}0, & z<0 \quad z>2,\\ \int_0^z 1dx, & 0<z<1,\\ \int_{z-1}^1 1dx, & 1<z<2,\end{cases}$$

$$f_Z(z) = \begin{cases} 0, & z < 0 \quad z > 2 \\ z, & 0 < z < 1 \\ 2 - z, & 1 < z < 2 \end{cases}$$



$$F_Z(z) = P(X + Y \le z)$$

$$= \iint_{x+y\le z} f(x, y)dxdy$$

$$z < 0$$

$$F_Z(z) = 0$$

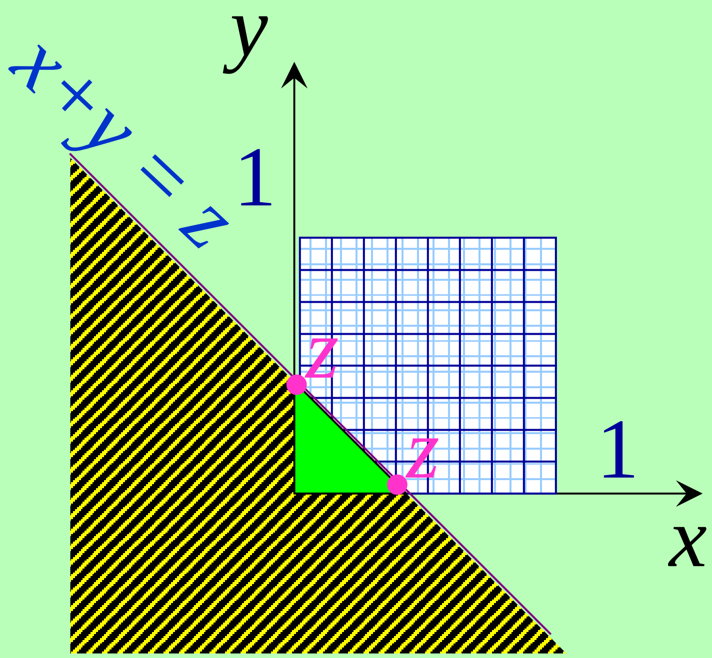
$$0 \le z < 1$$

$$F_Z(z) = \int_0^z dx \int_0^{z-x} 1 dy$$

$$= \int_0^z (z-x) dx$$

$$= z^2/2$$

$$\Rightarrow \quad f_Z(z) = z$$



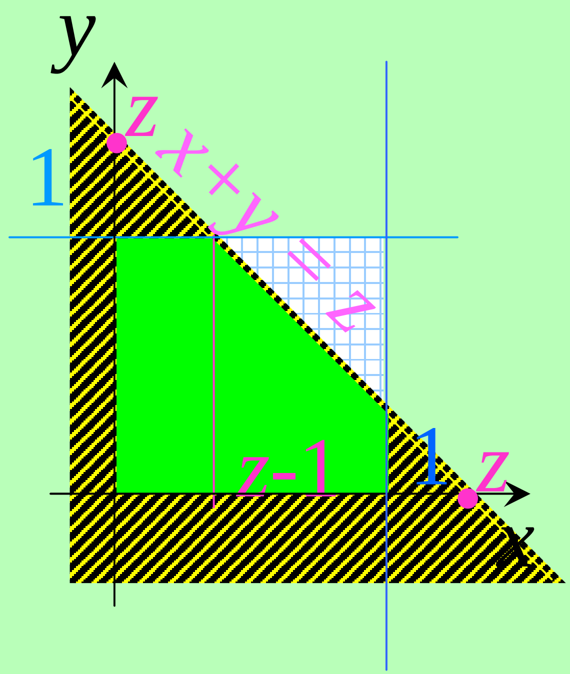
$$1 \le z < 2$$

$$F_Z(z) = (z-1) + \int_{z-1}^{1} dx \int_0^{z-x} 1 dy$$

$$= z - 1 + \int_{z-1}^{1} (z-x) dx$$

$$= 2z - z^2/2 - 1$$

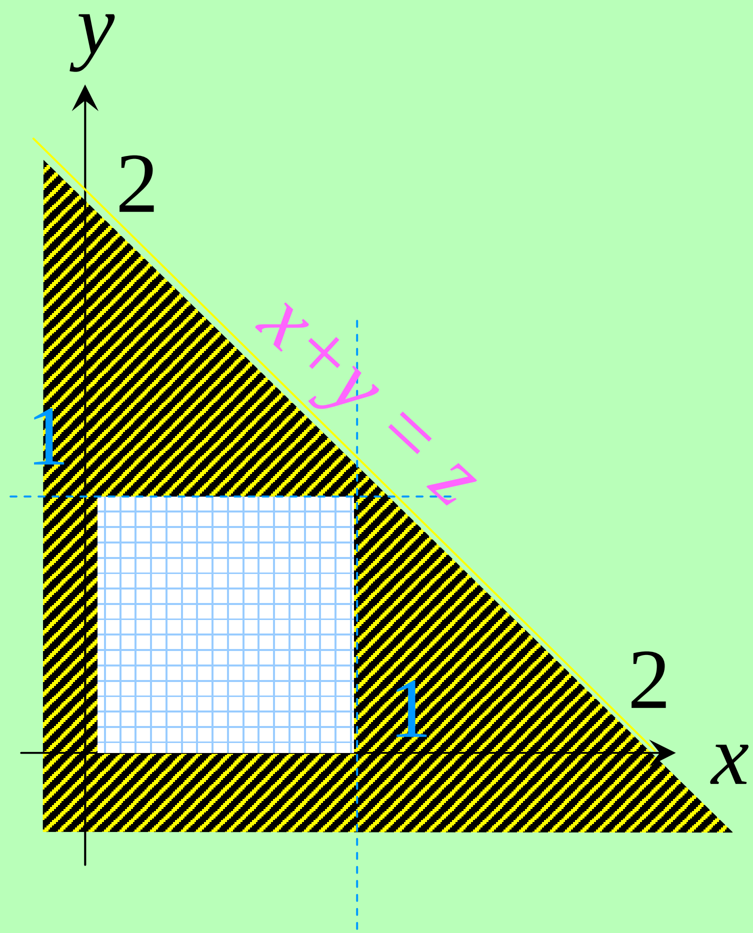
$$\Rightarrow f_Z(z) = 2 - z$$



$$2 \le z$$

$$F_Z(z) = 1$$

$$f_Z(z) = 0$$



$$f_Z(z) = \begin{cases} 0, & z<0 \text{或} z>2 \\ z, & 0<z<1 \\ 2-z, & 1<z<2 \end{cases}$$

例3 (X, Y) $d.f.$

$$f(x,y)=\begin{cases}3x, & 0<x<1, 0<y<x \\ 0, & \text{其他}\end{cases}$$

$Z = X + Y$, $f_Z(z)$

解法一 图形定限法

1 $$f_Z(z)=\int_{-\infty}^{+\infty} f(x, z-x)dx$$

$$f(x, z-x)=\begin{cases}3x, & 0<x<1, x<z<2x \\ 0, & \text{其他}\end{cases}$$

$z < 0 \qquad z > 2$,

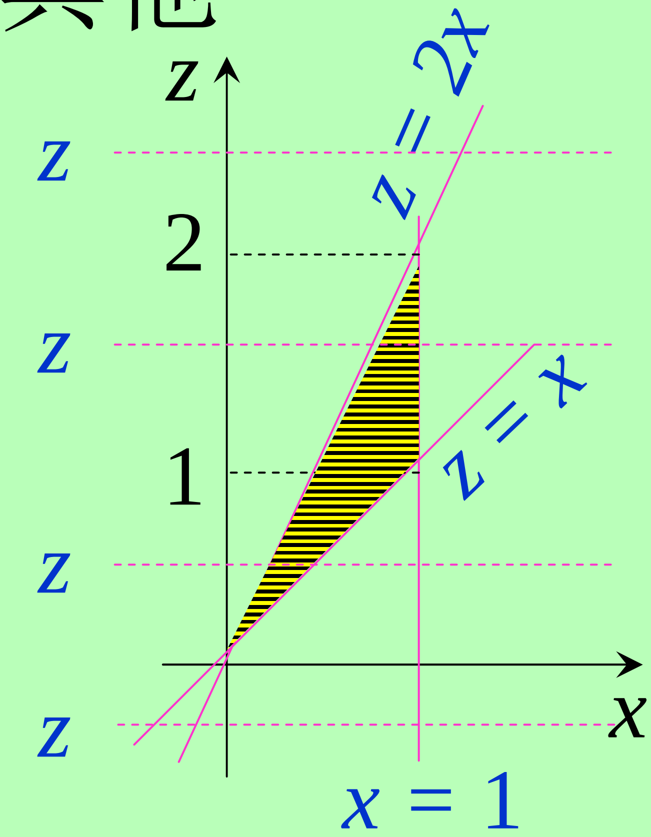
$$f_Z(z) = 0$$

$0 \le z < 1$,

$$f_Z(z)=\int_{z/2}^{z} 3x dx=\frac{9}{8}z^2$$

$1 \le z < 2$,

$$f_Z(z) = \int_{z/2}^{1} 3x dx = \frac{3}{2}(1-\frac{z^2}{4})$$



$$f_Z(z)=\begin{cases}\dfrac{9}{8}z^2, & 0\le z<1\\ \dfrac{3}{2}(1-\dfrac{z^2}{4}), & 1\le z<2\\ 0, & \text{其他}\end{cases}$$

不等式组定限法

$$f_Z(z)=\int_{-\infty}^{+\infty} f(x,z-x)dx$$

$$\begin{cases} 0<x<1 \\ 0<z-x<x \end{cases} \Rightarrow \begin{cases} 0<x<1 \\ \frac{z}{2}<x<z \end{cases} \quad (*)$$

,　　z

0　1　2

$z<0$　　$z\geq 2$　　$(*)$

$0\leq z<1$　　$(*)$　　$\frac{z}{2}<x<z$

$1\leq z<2$　　$(*)$　　$\frac{z}{2}<x<1$

$$f_Z(z)=\begin{cases}\int_{z/2}^{z}3x\,dx=\frac{9}{8}z^2 & 0\le z<1\\ \int_{z/2}^{1}3x\,dx=\frac{3}{2}(1-\frac{z^2}{4}) & 1\le z<2\\ 0 & \text{其他}\end{cases}$$

正态随机变量的结论

- X, Y , $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

推广 $X_1, X_2, \cdots, X_n$

$$X_i \sim N(\mu_i, \sigma_i^2), i = 1, 2, \cdots, n$$

$$\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

- $(X, Y) \sim N(\mu_1, \sigma_1^2; \mu_2, \sigma_2^2; \rho)$

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$

补充作业

(X, Y) $d.f.\ f(x,y)$

$Z = aX + bY + c$ $d.f.,$

a,b,c $a\,,\,b \neq 0$

$$f_Z(z) = \frac{1}{|b|}\int_{-\infty}^{+\infty} f\left(x, \frac{z-ax-c}{b}\right)dx \quad -\infty < z < \infty \quad (a.e.)$$

$$f_Z(z) = \frac{1}{|a|}\int_{-\infty}^{+\infty} f\left(\frac{z-by-c}{a}, y\right)dy \quad -\infty < z < \infty \quad (a.e.)$$

（证明见后面附录）

另一种计算 $f_Z(z)$ 的方法
——随机变量代换法

- r.v. (Z, V),

$$\begin{cases} Z = g(X, Y) \\ V = r(X, Y) \end{cases}$$

- (Z, V) d.f. $f(z, v)$

- $f_Z(z)$

求 (Z, V) 的 $p.d.f.$ $f_{ZV}(z, v)$ 的公式

(X, Y) $d.f.$ $f_{XY}(x, y)$

$$\begin{cases} z = g(x, y) \\ v = r(x, y) \end{cases}$$

唯一

$$\begin{cases} x = h(z, v) \\ y = s(z, v) \end{cases}$$

h, s ,

$$J = \begin{vmatrix} \dfrac{\partial h}{\partial z} & \dfrac{\partial h}{\partial v} \\ \dfrac{\partial s}{\partial z} & \dfrac{\partial s}{\partial v} \end{vmatrix}$$

$$f_{ZV}(z,v) = f_{XY}[h(z,v), s(z,v)]\,|J|$$

证 $F_{ZV}(z,v)=P(Z\le z,V\le v)$

$$=P(g(X,Y)\le z,r(X,Y)\le v)$$

$$=\iint\limits_{\substack{g(x,y)\le z\\ r(x,y)\le v}} f_{XY}(x,y)dxdy$$

$$=\int_{-\infty}^{z}\int_{-\infty}^{v} f_{XY}[h(z,v),s(z,v)]|J|dvdz$$

$$f_{ZV}(z,v)=f_{XY}[h(z,v),s(z,v)]|J|$$

(2) $Z = X / Y$

(X, Y) d.f. $f(x,y)$

$Z = X/Y$, $f_Z(z)$

$$\begin{cases} Z = X/Y \\ V = Y \end{cases} \Rightarrow \begin{cases} X = ZV \\ Y = V \end{cases} \Rightarrow |J| = \left| \begin{vmatrix} v & z \\ 0 & 1 \end{vmatrix} \right| = |v|$$

$$f_{ZV}(z,v) = f\left(zv, v\right)|v|$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZV}(z,v)dv = \int_{-\infty}^{+\infty} f(zu,v)|v|dv$$

例4 (X, Y)

$$F(x,y)=\begin{cases}1-e^{-x}-e^{-y}+e^{-(x+y)}, & x>0, y>0\\ 0, & \end{cases}$$

$Z = X / Y$ $p.d.f.$

解

$$f(x,y)=\begin{cases}e^{-(x+y)}, & x>0, y>0\\ 0, & \end{cases}$$

$$f_Z(z)=\int_{-\infty}^{+\infty} f(zv,v)\,|v|\,dv$$

$$f(zv,v)=\begin{cases}e^{-(zv+v)}, & z>0, v>0\\ 0, & \text{其他}\end{cases}$$

(3) **: $Z = X^2 + Y^2$**

(X, Y) d.f. $f(x,y)$

$$F_Z(z) = P(X^2 + Y^2 \le z)$$

$$= \begin{cases} 0, & z < 0, \\ \iint_{x^2+y^2\le z} f(x,y)\,dx\,dy & z \ge 0, \end{cases}$$

$$= \begin{cases} 0, & z < 0, \\ \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} f(r\cos\theta, r\sin\theta)\,r\,dr, & z \ge 0, \end{cases}$$

$$f_Z(z)=\begin{cases} 0, & z<0, \\ \dfrac{1}{2}\displaystyle\int_0^{2\pi} f(\sqrt{z}\cos\theta, \sqrt{z}\sin\theta)d\theta, & z\ge 0, \end{cases}$$

$X \sim N(0,1),\ Y \sim N(0,1),\ X\ ,Y$

$Z = X^2+Y^2\ ,$

$$f_Z(z)=\begin{cases} 0, & z<0, \\ \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2\pi}}e^{-\frac{z\cos^2\theta}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{z\sin^2\theta}{2}} d\theta, & z\ge 0, \end{cases}$$

$$f_Z(z)=\begin{cases} 0, & z<0, \\ \dfrac{1}{2}e^{-\frac{z}{2}}, & z\ge 0, \end{cases}$$

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(4) ( )

例5 X, Y , 0.5

0-1 . $M = \max\{X, Y\}$

解

Y \ X (p_{ij})	1	0
1	0.25	0.25
0	0.25	0.25

$\max\{X, Y\}$	1	0
P	0.75	0.25

X, Y , $X \sim F_X(x)$,

$Y \sim F_Y(y)$, $M = \max\{X, Y\}$, $N = \min\{X, Y\}$,

M, N .

$$F_M(u) = P(\max\{X, Y\} \le u)$$

$$= P(X \le u, Y \le u)$$

$$= P(X \le u) P(Y \le u)$$

$$= F_X(u) F_Y(u)$$

$$
\begin{aligned}
F_N(v) &= P(\min\{X,Y\} \le v) \\
&= 1 - P(\min\{X,Y\} > v) \\
&= 1 - P(X > v, Y > v) \\
&= 1 - P(X > v)P(Y > v) \\
&= 1 - [1 - F_X(v)][1 - F_Y(v)].
\end{aligned}
$$

推广

$$X_1, X_2, \cdots, X_n$$

$$X_i \sim F_i(x_i), \quad i = 1,2,\cdots,n$$

$$M = \max\{X_1, X_2, \cdots, X_n\}$$

$$N = \min\{X_1, X_2, \cdots, X_n\}$$

$$F_M(u) = \prod_{i=1}^{n} F_i(u)$$

$$F_N(v) = 1 - \prod_{i=1}^{n} (1 - F_i(v))$$

例6 L n

， (1) ； (2)

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n $X_1, X_2, \cdots, X_n$

$X_i \sim E(\lambda), \quad i = 1, 2, \cdots, n$

4 , L

X $d.f.$

解

$$f_{X_i}(x_i) = \begin{cases} \lambda e^{-\lambda x_i}, & x_i > 0 \\ 0, & \end{cases}$$

$$F_{X_i}(x_i) = \begin{cases} 1 - e^{-\lambda x_i}, & x_i > 0 \\ 0, & \end{cases}$$

(1) $X = \min\{X_1, X_2, \cdots, X_n\}$

$$F_X(x) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(x))$$

$$1 - F_{X_i}(x) = \begin{cases} e^{-\lambda x}, & x > 0, \\ 1, & x \le 0 \end{cases}$$

$$f_X(x) = \begin{cases} n\lambda e^{-n\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(2) $X = \max\{X_1, X_2, \cdots, X_n\}$

$$F_X(x) = \prod_{i=1}^{n} F_{X_i}(x)$$

$$= \begin{cases} (1 - e^{-\lambda x})^n, & x > 0, \\ 0, & x \le 0 \end{cases}$$

$$f_X(x) = \begin{cases} n\lambda e^{-\lambda x}(1-e^{-\lambda x})^{n-1}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

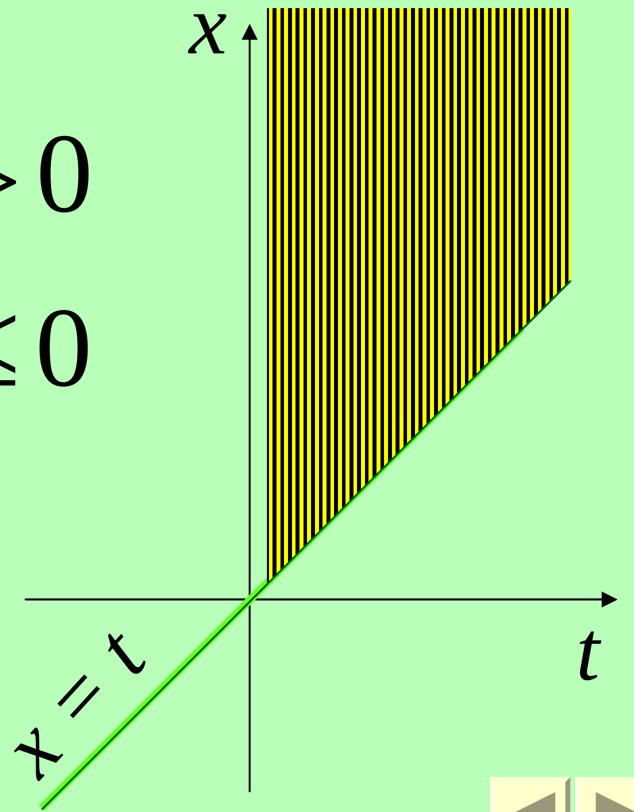
(3) $X = X_1 + X_2 + \cdots + X_n$

$n = 2$

$$f_{X_1+X_2}(x) = \int_{-\infty}^{+\infty} f_{X_1}(t) f_{X_2}(x-t)dt$$

$$= \begin{cases} \int_0^x e^{-\lambda t} e^{-\lambda(x-t)} dt, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$= \begin{cases} x e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$



, $X_1 + X_2$ X_3 ,

$$f_{X_1+X_2+X_3}(x) = \int_{-\infty}^{+\infty} f_{X_1+X_2}(t) f_{X_3}(x-t)\,dt$$

$$= \begin{cases} \int_0^x t e^{-\lambda t} e^{-\lambda(x-t)}\,dt, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$= \begin{cases} \dfrac{x^2}{2!} e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$f_X(x) = \begin{cases} \dfrac{x^{n-1}}{(n-1)!} e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(4) $F_X(x) = P(X \le x)$

$$= \begin{cases} 1 - P(X > x), & x \ge 0, \\ 0, & x < 0 \end{cases}$$

$P(X > x) = P\left(X_1, \cdots, X_n\text{中至少有 } k \text{ 个大于} x\right)$

$$= \sum_{j=k}^{n} C_n^j \left[P(X_1 > x)\right]^j \left[P(X_1 \le x)\right]^{n-j}$$

$$F_X(x) = \begin{cases} 1 - \sum_{j=k}^{n} C_n^j \left[e^{-\lambda x}\right]^j \left[1 - e^{-\lambda x}\right]^{n-j}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

$$\frac{d}{dx}\left[-\sum_{j=k}^{n} C_n^j \left[e^{-\lambda x})\right]^j \left[1-e^{-\lambda x}\right]^{n-j}\right]$$

$$= \sum_{j=k}^{n-1} C_n^j \lambda j e^{-\lambda j x}\left(1-e^{-\lambda x}\right)^{n-j} + n\lambda e^{-n\lambda x}$$

$$- \sum_{j=k}^{n-1} C_n^j \lambda (n-j) e^{-\lambda (j+1) x}\left(1-e^{-\lambda x}\right)^{n-j-1}$$

$$= \sum_{j=k}^{n} C_n^j \lambda j e^{-\lambda j x}\left(1-e^{-\lambda x}\right)^{n-j}$$

$$- \sum_{j=k}^{n-1} C_n^{j+1} (j+1) \lambda e^{-\lambda (j+1) x}\left(1-e^{-\lambda x}\right)^{n-j-1}$$

$$= \sum_{j=k}^{n} C_n^j \lambda j e^{-\lambda j x}\left(1-e^{-\lambda x}\right)^{n-j}$$

$$- \sum_{j=k+1}^{n} C_n^j j \lambda e^{-\lambda j x}\left(1-e^{-\lambda x}\right)^{n-j}$$

$$= C_n^k k\lambda e^{-k\lambda x}(1-e^{-\lambda x})^{n-k}$$

$$f_X(x) = \begin{cases} C_n^k k\lambda e^{-k\lambda x}(1-e^{-\lambda x})^{n-k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

P.134

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23	26
29	30

第9周

问 题

设随机变量 X 与 Y 相互独立，且

$$X \sim B(1, 0.6),\ Y \sim f(y).$$

求随机变量 $Z = 3X - Y$ 的概率密度函数 $g(z)$.

第10周　问 题

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附 录

二项分布可加性的证明

$X \quad Y \qquad ,$

$X \sim B(n, p) \quad Y \sim B(m, p),$

$$X + Y \sim B(n + m,\ p)$$

证 $Z = X + Y$

$0, 1, 2, \ldots,\ n + m$

$$\sum_{i=0}^{k} C_n^i C_m^{k-i} = C_{n+m}^k$$

$$P(Z=k)=\sum_{i=0}^{k}P(X=i,Y=k-i),$$

$$=\sum_{i=0}^{k}P(X=i)P(Y=k-i),$$

$$=\sum_{i=0}^{k}C_n^i p^i(1-p)^{n-i}C_m^{k-i}p^{k-i}(1-p)^{m-k+i}$$

$$=C_{n+m}^{k}p^{k}(1-p)^{n+m-k}$$

$$k=0,1,2,\ \ldots,\ n+m$$

$$X+Y\sim B(n+\mathrm{m},\ p)$$

证二 (1) $n \le \mathrm{m}$, $k \le n$

$$P(Z=k)=\sum_{i=0}^{k} P(X=i, Y=k-i),$$

$$=\sum_{i=0}^{k} P(X=i) P(Y=k-i),$$

$$=\sum_{i=0}^{k} C_n^i p^i (1-p)^{n-i} C_m^{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$=C_{n+m}^{k} p^{k} (1-p)^{n+m-k}$$

$$\sum_{i=0}^{k} C_n^i C_m^{k-i} = C_{n+m}^{k}$$

(2) $n < k \le m$

$$P(Z=k)=\sum_{i=0}^{n} P(X=i, Y=k-i)$$

$$=\sum_{i=0}^{n} C_n^i p^i (1-p)^{n-i} C_m^{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$= C_{n+m}^k p^k (1-p)^{n+m-k}$$

(3) $m < k \le n + \mathrm{m}$

$$P(Z = k) = \sum_{i=k-m}^{n} P(X = i, Y = k - i)$$

$$= \sum_{i=k-m}^{n} C_n^i p^i (1-p)^{n-i} C_m^{k-i} p^{k-i} (1-p)^{m-k+i}$$

$$= C_{n+m}^{k} p^{k} (1 - p)^{n+m-k}$$

$$X + Y \sim B(n + \mathrm{m},\ p)$$

由二项分布背景, 不难理解$X+Y$表示做了$n+m$次试验, 事件A发生的次数.

前例3

$$(X, Y)$$

$$f(x,y)=\begin{cases} 3x, & 0<x<1, 0<y<x \\ 0, & \text{其他} \end{cases}$$

$$Z = X + Y, \quad f_Z(z)$$

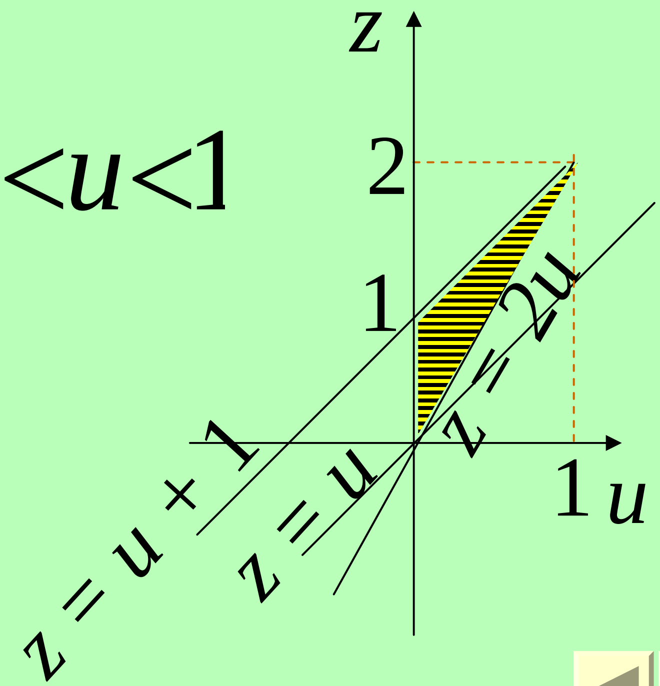
$$\begin{cases} Z = X + Y \\ U = \quad Y \end{cases} \longrightarrow \begin{cases} X = Z - U \\ Y = \quad U \end{cases}$$

$$|J| = \left| \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \right| = 1$$

$$f_{ZU}(z,u) = f(z-u,u)\cdot 1$$

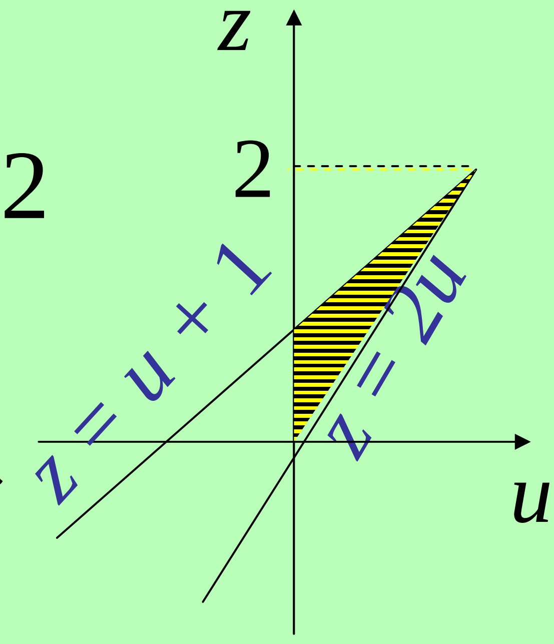
$$= \begin{cases} 3(z-u), & 0<z-u<1, 0<u<z-u \\ 0, & \text{其他} \end{cases}$$

$$= \begin{cases} 3(z-u), & 2u<z<u+1, 0<u<1 \\ 0, & \end{cases}$$



$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZU}(z,u)\,du$$

$$= \begin{cases} \int_0^{z/2} 3(z-u)\,du = \frac{9}{8}z^2, & 0 \le z < 1 \\ \int_{z-1}^{z/2} 3(z-u)\,du = \frac{3}{2}\left(1-\frac{z^2}{4}\right), & 1 \le z < 2 \\ 0, & \text{其他} \end{cases}$$



附例 (X, Y)

$$f(x,y)=\begin{cases}3x, & 0<x<1, 0<y<x\\ 0, & \text{其他}\end{cases}$$

$Z = 3X - 2Y$, $f_Z(z)$

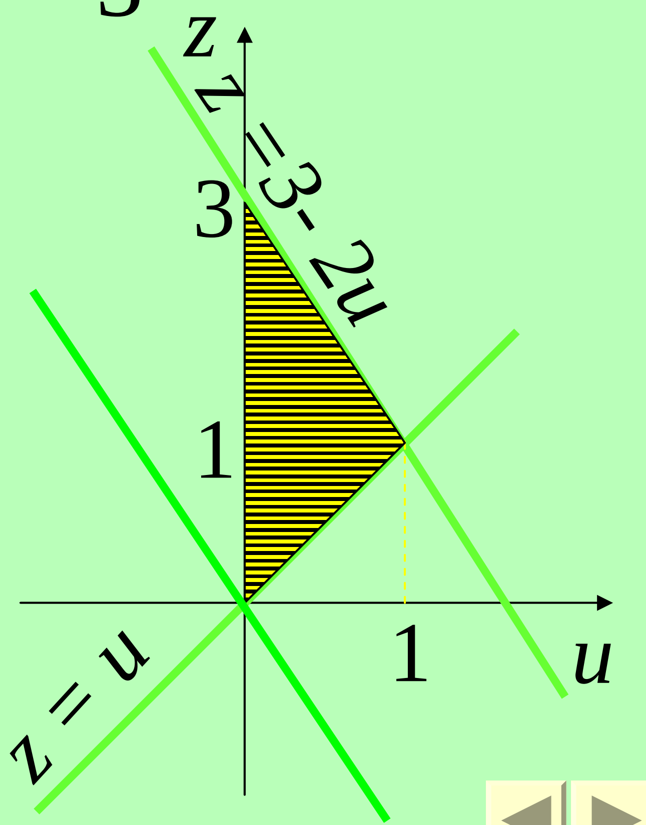
解 $\begin{cases}Z = 3X - 2Y\\ U = Y\end{cases} \Rightarrow \begin{cases}X = \frac{1}{3}(Z+2U)\\ Y = U\end{cases}$

$$|J| = \left|\begin{matrix}1/3 & 2/3\\ 0 & 1\end{matrix}\right| = \frac{1}{3}$$

$$f_{ZU}(z,u) = f\left(\frac{1}{3}(z+2u), u\right)\cdot\frac{1}{3}$$

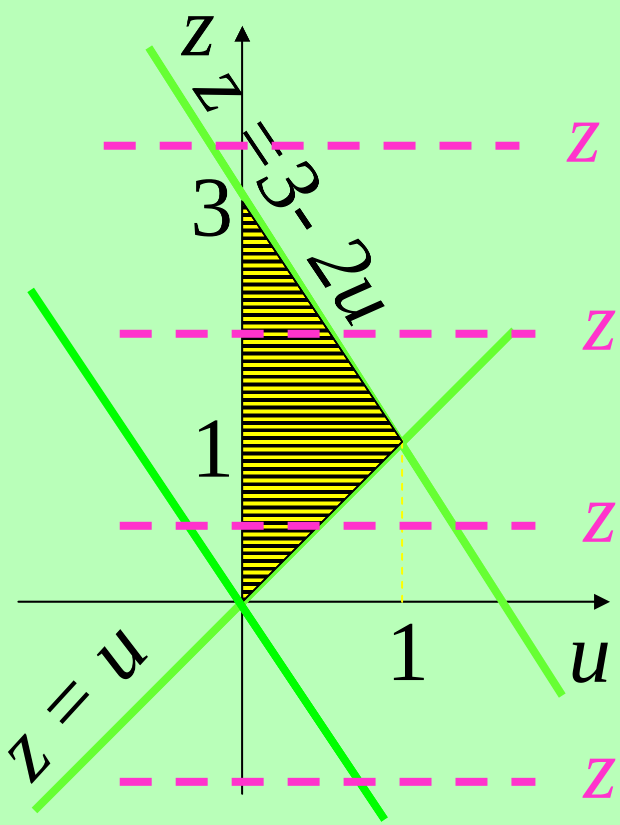
$$= \begin{cases} \frac{1}{3}(z+2u), & 0<\frac{1}{3}(z+2u)<1, 0<u<\frac{1}{3}(z+2u) \\ 0, & \end{cases}$$

$$= \begin{cases} \frac{1}{3}(z+2u), & u<z<3-2u, 0<u<1 \\ 0, & \end{cases}$$



$$f_Z(z)=\int_{-\infty}^{+\infty} f_{ZU}(z,u)\,du$$

$$=\begin{cases} \int_0^z \frac{1}{3}(z+2u)\,du=\frac{2}{3}z^2, & 0<z<1 \\ \int_0^{\frac{3-z}{2}} \frac{1}{3}(z+2u)\,du=\frac{1}{12}\left(9-z^2\right), & 1<z<3 \\ 0, & 其他 \end{cases}$$



附例 (X, Y) $f(x, y)$

$Z = aX + bY + c$,

a, b, c $a, b \neq 0$

$$\begin{cases} Z = aX + bY + c \\ U = Y \end{cases} \Rightarrow \begin{cases} X = \dfrac{Z - bU - c}{a} \\ Y = U \end{cases}$$

$$|J| = \begin{vmatrix} \dfrac{1}{a} & -\dfrac{b}{a} \\ 0 & 1 \end{vmatrix} = \frac{1}{|a|}$$

$$f_{ZU}(z,u) = f\left(\frac{z-bu-c}{a}, u\right)\frac{1}{|a|}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f\left(\frac{z-bu-c}{a}, u\right)\frac{1}{|a|}du$$