《概率统计》

第六章习题课

6-1(3)

$$E(\overline{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{\theta+1}{\theta+2} \checkmark$$

$$D(\overline{X}) = \underline{\underline{E(\overline{X}^2)}} - (E\overline{X})^2 \overset{?}{=} \underline{\underline{\frac{\theta+1}{\theta+3}}} - \left(\frac{\theta+1}{\theta+2}\right)^2$$

正确求解

$$D(\overline{X}) = \frac{\sigma^2}{n} = \frac{D(X_i)}{n} = \frac{1}{n}\left[EX_i^2 - (EX_i)^2\right]$$

$$= \frac{1}{n}\left[\frac{\theta+1}{\theta+3} - \frac{(\theta+1)^2}{(\theta+2)^2}\right] = \frac{\theta+1}{n(\theta+3)(\theta+2)^2}$$

仿P.191~192 求得 $E(S^2)$ ~~=~~ σ^2

σ^2 与题无关，等于没求！

正确求解

$$E(S^2)=\sigma^2=D(X_i)=\frac{\theta+1}{(\theta+3)(\theta+2)^2}$$

6-2 **(1)**

$$\Omega=\left\{(x_1,\cdots,x_n)\;\middle|\; \begin{array}{l} x_i \overset{?}{=} C_N^k p^k(1-p)^{N-k}, \\ k=0,1,\cdots,N \end{array}\right\}$$

x_i 怎能取两项概率?

正确求解

(1) $\Omega=\left\{(x_1,\cdots,x_n)\;\middle|\;\begin{matrix} x_i=0,1,\cdots,N \\ i=1,2,\cdots,n\end{matrix}\right\}$

k 是什么？

再看

(2) $F(x_1,\cdots,x_{10})=\prod_{i=1}^{10} C_N^{x_i}\, p^{\cancel{k}}(1-p)^{N-\cancel{k}}\,\sigma^2$

(3) $E(\overline{X})=\frac{1}{n}\sum_{i=1}^{n}E(X_i)=\frac{1}{n}\sum_{i=1}^{n}\cancel{n}p=np$

$E(S^2)=\cancel{n}p(1-p)$

注意二项分布的参数是什么？

正确求解

(2) $$P(X_1 = x_1, \cdots, X_{10} = x_{10})$$

$$= \prod_{i=1}^{10} C_N^{x_i} p^{x_i} (1-p)^{N-x_i}$$

$$= p^{\sum_{i=1}^{10} x_i} (1-p)^{10N - \sum_{i=1}^{10} x_i} \prod_{i=1}^{10} C_N^{x_i}$$

$$x_i = 0, 1, \cdots, N, \quad i = 1, 2, \cdots, 10.$$

(3) $$E(\overline{X}) = N p$$

$$E(S^2) = N p (1-p)$$

6-3

(1) $\Omega=\left\{\left(x_1,\cdots,x_n\right)\middle|\, x_i\in \xcancel{X},\ i=1,2,\cdots,n\right\}$

x_i 岂能属于随机变量！

正确求解

(1) $\Omega=\left\{\left(x_1,\cdots,x_n\right)\middle|\, x_i\in R,\ i=1,2,\cdots,n\right\}$

(2)
$$L(x_1,\cdots,x_n)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$
$$=\frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n}\cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

(3) $E(\overline{X})=\mu,\quad D(\overline{X})=\dfrac{\sigma^2}{n},\quad E(S^2)=\sigma^2$

6-6

利用P.200推论2， 求得

$$S_W=\sqrt{\frac{(10-1)\cdot\cancel{3}+(15-1)\cdot\cancel{3}}{10+15-2}}=\sqrt{3}$$

$$P(\left|\overline{X}-\overline{Y}\right|>0.3)=P\left(\left|\frac{\overline{X}-\overline{Y}}{S_W\sqrt{\frac{1}{10}+\frac{1}{15}}}\right|>\frac{0.3}{S_W\sqrt{\frac{1}{10}+\frac{1}{15}}}\right)$$

$$\triangleq P(\left|T(23)\right|>0.3\sqrt{2})=P(\left|T(23)\right|>T_{\frac{\alpha}{2}}(23))=\alpha$$

$$\longrightarrow\ T_{\frac{\alpha}{2}}(23)=0.3\sqrt{2}=0.424\ \longrightarrow\ \alpha=0.6744$$

书后表中查不到此数

本题未必有 $S_1^2=S_2^2=3$

故不能用 t 分布

正确求解

由题设得 $\overline{X} \sim N(20, 0.3)$, $\overline{Y} \sim N(20, 0.2)$

$$\longrightarrow \overline{X} - \overline{Y} \sim N(0, 0.5)$$

$$P(\left|\overline{X} - \overline{Y}\right| > 0.3) = 1 - P(\left|\overline{X} - \overline{Y}\right| \le 0.3)$$

$$= 1 - P(-0.3 \le \overline{X} - \overline{Y} \le 0.3)$$

$$= 1 - \Phi(0.3\sqrt{2}) + \Phi(-0.3\sqrt{2})$$

$$= 2 - 2\Phi(0.3\sqrt{2}) = 0.6744$$

6-7

$$F_{X_{(1)}}(z)=1-[1-F(z)]^n,\quad F(x)=1-e^{-\lambda x},x\ge 0$$

$$f_{X_{(1)}}(z)=F'_{X_{(1)}}(z)=n\lambda e^{-n\lambda z},z\ge 0$$

$$\therefore E(X_{(1)})=\int_0^\infty z f_{X_{(1)}}(z)dz=\int_0^\infty n\lambda z e^{-n\lambda z}=\frac{1}{n\lambda}$$

$$F_{X_{(n)}}(z)=[F(z)]^n,\quad f_{X_{(n)}}(z)=n\lambda(1-e^{-\lambda z})^{n-1}e^{-\lambda z},z\ge 0$$

$$\therefore E(X_{(n)})=\int_0^\infty z f_{X_{(n)}}(z)dz=\int_0^\infty n\lambda z(1-e^{-\lambda z})^{n-1}e^{-\lambda z}dz$$

$$=\int_0^\infty n\lambda z(e^{-\lambda z}-C_{n-1}^1 e^{-2\lambda z}+C_{n-1}^2 e^{-3\lambda z}+\cdots$$

$$+(-1)^{n-2}C_{n-1}^{n-2}e^{-(n-1)\lambda z}+(-1)^{n-1}e^{-n\lambda z})dz$$

$$=\frac{1}{\lambda}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=\sum_{k=1}^{n}\frac{1}{\lambda k}$$

或者

$$E(X_{(n)})=\int_0^{\infty} n\lambda\sum_{k=0}^{n-1}C_{n-1}^{k}(-e^{-\lambda z})e^{-\lambda z}z\,dz$$

$$=n\lambda\sum_{k=0}^{n-1}C_{n-1}^{k}(-1)^{k}\int_0^{\infty} ze^{-\lambda(k+1)z}dz$$

$$=n\sum_{k=0}^{n-1}C_{n-1}^{k}(-1)^{k}\frac{1}{\lambda(k+1)^{2}}=\sum_{k=1}^{n}C_{n}^{k}(-1)^{k-1}\frac{1}{\lambda k}$$

这两个答案是相等的

补充作业

设 $X_1, X_2, \cdots, X_{2n} (n \geq 2)$ 为从正态总体 $X \sim N(\mu, \sigma^2)$ 中抽取的简单随机样本

其样本均值为 $\overline{X} = \frac{1}{2n}\sum_{i=1}^{2n} X_i$ 求统计量

$Y = \sum_{i=1}^{n}(X_i + X_{n+i} - 2\overline{X})^2$ 的期望 $(\sigma > 0)$.

本题各种答案

$$E(Y) = \begin{cases} 2n\sigma^2 & \text{—— 3503班} \\ (n-1)\sigma^2 & \text{—— 3407班} \\ 4(n-1)\sigma^2 & \text{—— 3406班} \\ (2n-1)\sigma^2 + 2\mu^2 & \text{—— 3403班} \\ 4n\sigma^2 + 6(n-1)\mu^2 & \text{—— 3404班} \end{cases}$$

解一

令 $2Z_i = X_i + X_{n+i} \Rightarrow \bar{Z} = \bar{X} = \dfrac{1}{2n}\sum_{i=1}^{2n} X_i$

$$Y = \sum_{i=1}^{n}(2Z_i - 2\bar{Z})^2$$

$$= 4\sum_{i=1}^{n}(Z_i - \bar{Z})^2 \overset{?}{\sim} 4\chi^2(n-1)$$

$$\Rightarrow E(Y) = 4(n-1).$$

解一

$$2Z_i = X_i + X_{n+i} \Rightarrow \bar{Z} = \bar{X} = \frac{1}{2n}\sum_{i=1}^{2n} X_i$$

$$Z_i \sim N(\mu, \sigma^2/2) \quad Y = \sum_{i=1}^{n}(2Z_i - 2\bar{Z})^2$$

$$Y/(2\sigma^2) = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(2Z_i - 2\bar{Z})^2$$

$$= \sum_{i=1}^{n}(Z_i - \bar{Z})^2 \Big/ (\sigma^2/2) \sim \chi^2(n-1)$$

所以 $E(Y) = 2(n-1)\sigma^2.$

解二

令 $Z_i = X_i + X_{n+i} - 2\bar{X}$, $E(Z_i) = 0$, $i = 1 \sim n$

$$D(\bar{X}) = \sigma^2 / 2n$$

$$E(Z_i^2) = E^2(Z_i) + D(Z_i) = D(Z_i)$$

$$= D(X_i + X_{n+i} - 2\bar{X})$$

$$\overset{?}{=} D(X_i) + D(X_{n+i}) + 4D(\bar{X})$$

$$= 2(n+1)\sigma^2 / n$$

故 $$E(Y) = E\left(\sum_{i=1}^{n} Z_i^2\right) = \sum_{i=1}^{n} E(Z_i^2)$$

$$= 2(n+1)\sigma^2.$$

解二 令 $Z_i = X_i + X_{n+i} - 2\bar{X}$, $E(Z_i) = 0$, $i = 1 \sim n$

$$Z_i = \frac{1}{n}[-X_1 - X_2 - \cdots + (n-1)X_i$$

$$- X_{i+1} - \cdots + (n-1)X_{n+i} - \cdots - X_{2n}]$$

$$E(Z_i^2) = E^2(Z_i) + D(Z_i) = D(Z_i)$$

$$= [2n - 2 + 2(n-1)^2]\sigma^2 / n^2$$

$$= 2(n-1)\sigma^2 / n$$

所以 $E(Y) = E\left(\sum_{i=1}^{n} Z_i^2\right) = \sum_{i=1}^{n} E(Z_i^2) = 2(n-1)\sigma^2.$

解三　令 $Z_i = X_i + X_{n+i}$　则 $Z_i \sim N(2\mu, 2\sigma^2)$

$$\therefore \quad \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{n}\sum_{i=1}^{n}(X_i + X_{n+i}) = 2\bar{X}.$$

则　$$Y = \sum_{i=1}^{n}(Z_i - \bar{Z})^2$$

$$\frac{1}{n-1}E(Y) = E\left(\frac{1}{n-1}Y\right) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2\right)$$

$$= D(Z_i) = 2\sigma^2$$

所以　$E(Y) = 2(n-1)\sigma^2.$