§1.3 导数

1.3.1 导数的定义

定义：设函数 $w = f(z)$ 定义于区域 B 上，$z_0 \in B, z_0 + \Delta z \in B(\Delta z = \Delta x + \mathrm{i}\Delta y)$，如果当 Δz 按任意方式趋于0时存在极限

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

则称 $w = f(z)$ 在点 z_0 可导（可微），此极限称为函数 $f(z)$ 在 z_0 点的导数（微商）.

1.3.2 求导公式

复变函数导数与微分的定义在形式上与高等数学中一元函数导数与微分的定义一致，因而实变函数理论中关于导数的运算规则和公式

往往可应用于复变函数. 例如

$$\begin{cases} \frac{d}{dz}(w_1 \pm w_2) = \frac{dw_1}{dz} \pm \frac{dw_2}{dz}, \\ \frac{d}{dz}(w_1 w_2) = \frac{dw_1}{dz} w_2 + w_1 \frac{dw_2}{dz}, \\ \frac{d}{dz}\left(\frac{w_1}{w_2}\right) = \frac{w_1' w_2 - w_1 w_2'}{w_2^2}, \\ \frac{dw}{dz} = 1/\frac{dz}{dw}, \\ \frac{d}{dz}F(w) = \frac{dF}{dz} \cdot \frac{dw}{dz}; \end{cases} \qquad \begin{cases} \frac{d}{dz} z^n = n z^{n-1}, \\ \frac{d}{dz} e^z = e^z, \\ \frac{d}{dz} \sin z = \cos z, \\ \frac{d}{dz} \cos z = -\sin z, \\ \frac{d}{dz} \ln z = \frac{1}{z}. \end{cases}$$

1.3.3 复变函数可导的必要条件

必须指出，复变函数和实变函数的导数定义，虽然形式上一样度质上却有很大的不同. 这是因为实变数 Δx 只能沿着实独逼近零、复变数 Δz 却可以沿复数平面上的任一曲线逼近零. 因此，跟实变函数的可导相比，复变函数的可导是一种严格得多的要求.

让复变函数沿实轴和虚轴两个特殊方向趋于零，且极限相等，可以得到函数可导的必要条件.

$$\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)+\mathrm{i}v(x+\Delta x,y)-u(x,y)-\mathrm{i}v(x,y)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left\{\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+\mathrm{i}\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}\right\}$$

$$=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial u}{\partial y}. \qquad (1.3\text{-}1)$$

$$\lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)+\mathrm{i}v(x,y+\Delta y)-u(x,y)-\mathrm{i}v(x,y)}{\mathrm{i}\Delta y}$$

$$=\lim_{\Delta y\to 0}\left\{\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}-\mathrm{i}\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}\right\}$$

$$=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial u}{\partial y}. \qquad (1.3\text{-}2)$$

$$\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial u}{\partial y}.$$

由此得复变函数可导的必要条件—Cauchy-Riemann 条件（方程），简称 C-R 条件.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \tag{1.3-3}$$

1.3.4 C-R 条件的极坐标形式

由

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \longrightarrow \begin{cases} \dfrac{\partial y}{\partial \varphi} = \rho\cos\varphi \\ \dfrac{\partial x}{\partial \rho} = \cos\varphi \end{cases} \longrightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \rho}\dfrac{\partial \rho}{\partial x} = \dfrac{\partial v}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial \rho}\dfrac{\partial \rho}{\partial x} = -\dfrac{\partial u}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial u}{\partial y} \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{\partial u}{\partial \rho} = \dfrac{\partial v}{\partial \varphi}\dfrac{\partial \varphi}{\partial y}\dfrac{\partial x}{\partial \rho} = \dfrac{1}{\rho}\dfrac{\partial v}{\partial \varphi} \\ \dfrac{\partial v}{\partial \rho} = -\dfrac{\partial u}{\partial \varphi}\dfrac{\partial \varphi}{\partial y}\dfrac{\partial x}{\partial \rho} = -\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi} \end{cases} \rightarrow \begin{cases} \dfrac{\partial u}{\partial \rho} = \dfrac{1}{\rho}\dfrac{\partial v}{\partial \varphi} \\ \dfrac{\partial v}{\partial \rho} = -\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi} \end{cases}$$

满足 C-R 条件的函数不一定可导（因为它们只能保证函数沿实轴和虚轴有相同的极限）.

1.3.5 复变函数可导的充分必要条件

可以证明：函数 $f(z)$ 可导的充分必要条件是，函数 $f(z)$ 的偏导数 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ 和 $\frac{\partial v}{\partial y}$ 存在，连续，且满足 C-R 条件.

复变函数的实部和虚部通过 $C-R$ 条件紧密联系起来.

复变函数 $f(z)$ 在点 $z = x + \mathrm{i}y$ 的导数可表示为

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x}$$

作业(No.1)

P. 5：1(1)—1(3)；2(1)—2(3)、2(7)、3(1)—3(3)、3(5)

P. 9：2(1)、2(5)、2(6)

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